VIRILE AGITUR

Teacher's Name ____________________

Student Number

Knox Grammar School

2013

Trial Higher School Certificate Examination

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time - 3 hours
- Write using blue or black pen only
- Board approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Subject Teachers
Mr I Bradford
Mr M Vuletich

Setter

Mr Vuletich

This paper MUST NOT be removed from the examination room

Number of Students in Course: 31

Total Marks – 100

Section I 10 Marks

- Answer Questions 1 to 10
- Use the Multiple Choice Answer Sheet

Section II 90 Marks

- Answer Questions 11 to 16
- All questions are worth 15 marks
- Answer each question in a separate Writing Booklet.

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Section I

10 Marks
Attempt Questions 1–10
Allow about 20 minutes for this section

Use the multiple-choice answer sheet for Questions 1-10

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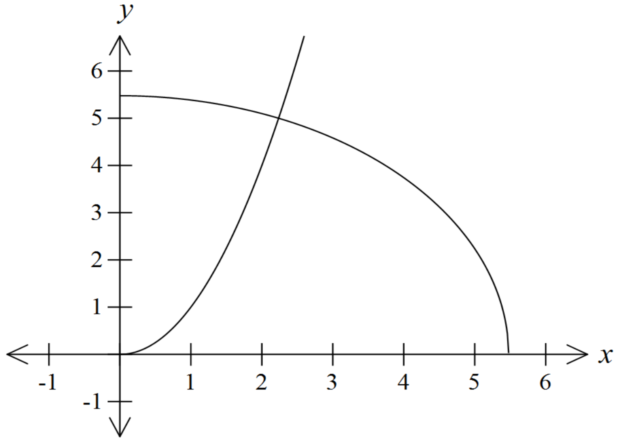
1 Let $z = 1 + 2i$ and $w = -2 + i$. What is the value of $\dfrac{5}{iw}$?

(A) $-1 - 2i$
(B) $-1 + 2i$
(C) $1 - 2i$
(D) $1 + 2i$

2 What is the volume of the solid formed when the region bounded by the curves, $y = x^2$, $y = \sqrt{30 - x^2}$ and the y-axis is rotated about the y-axis?



What is the correct expression for volume of this solid using the method of cylindrical shells?

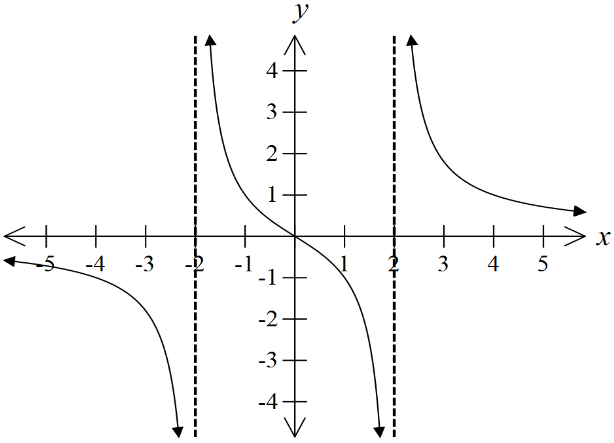
(A) $V = \int_0^{\sqrt{5}} 2\pi\left(x^2 - \sqrt{30 - x^2}\right)dx$

(B) $V = \int_0^{\sqrt{5}} 2\pi x\left(x^2 - \sqrt{30 - x^2}\right)dx$

(C) $V = \int_0^{\sqrt{5}} 2\pi\left(\sqrt{30 - x^2} - x^2\right)dx$

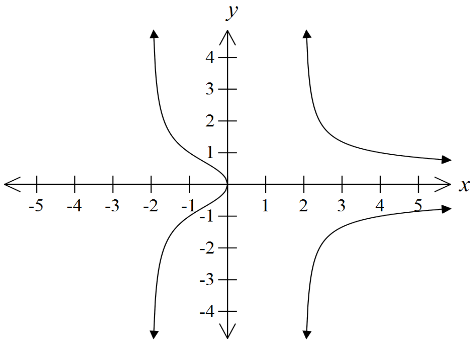
(D) $V = \int_0^{\sqrt{5}} 2\pi x\left(\sqrt{30 - x^2} - x^2\right)dx$

3 The diagram shows the graph of the function $y = f(x)$.

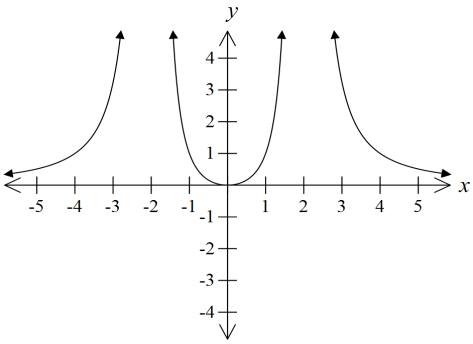


Which of the following is the graph of $y^2 = f(x)$?

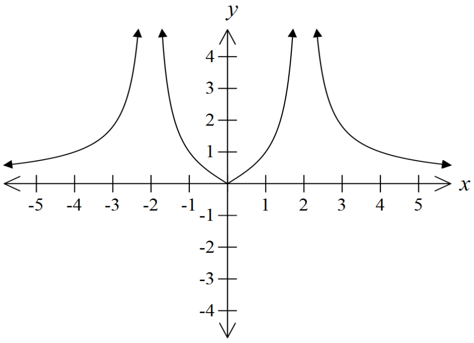
(A)



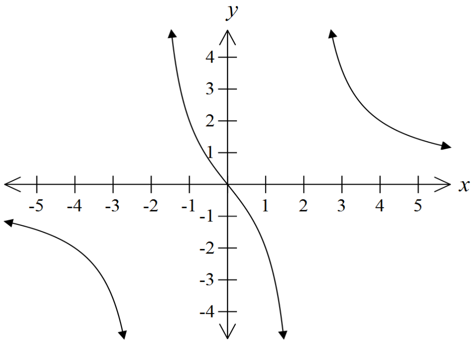
(B)



(C)



(D)



4 Let α, β and γ be roots of the equation $x^3 + 3x^2 + 4 = 0$. Which of the following polynomial equations have roots α^2, β^2 and γ^2?

(A) $x^3 - 9x^2 - 24x - 4 = 0$

(B) $x^3 - 9x^2 - 12x - 4 = 0$

(C) $x^3 - 9x^2 - 24x - 16 = 0$

(D) $x^3 - 9x^2 - 12x - 16 = 0$

5 A particle of mass *m* is moving in a straight line under the action of a force.

$$F = \frac{m}{x^3}(6 - 10x)$$

What of the following is an expression for its velocity in any position, if the particle starts from rest at $x = 1$?

(A) $v = \pm\frac{1}{x}\sqrt{(-3 + 10x - 7x^2)}$

(B) $v = \pm x\sqrt{(-3 + 10x - 7x^2)}$

(C) $v = \pm\frac{1}{x}\sqrt{2(-3 + 10x - 7x^2)}$

(D) $v = \pm\frac{1}{x}\sqrt{2(-3 + 10x + 7x^2)}$

6 Which of the following is an expression for $\int \frac{2}{x^2 + 4x + 13} dx$?

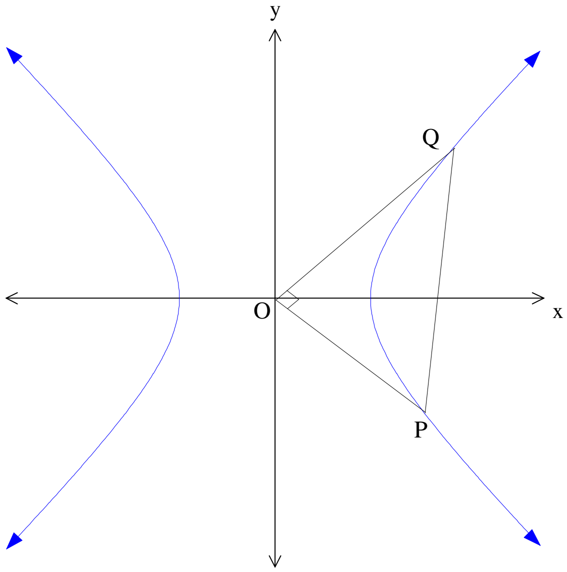
(A) $\frac{1}{3}\tan^{-1}\frac{(x+2)}{3} + c$	(B) $\frac{2}{3}\tan^{-1}\frac{(x+2)}{3} + c$
(C) $\frac{1}{9}\tan^{-1}\frac{(x+2)}{9} + c$	(D) $\frac{2}{9}\tan^{-1}\frac{(x+2)}{9} + c$

7 Consider the hyperbola with the equation $\frac{x^2}{16} - \frac{y^2}{9} = 1$.

What are the coordinates of the foci of the hyperbola?

(A) $(\pm 4, 0)$	(B) $(0, \pm 4)$
(C) $(0, \pm 5)$	(D) $(\pm 5, 0)$

8 The diagram below shows the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ where $a>b>0$. The points $P(a\sec\theta, b\tan\theta)$ and $Q(a\sec\alpha, b\tan\alpha)$ lie on the hyperbola and the chord PQ subtends a right angle at the origin.



Use the parametric representation of the hyperbola to determine which of the following expressions is correct?

(A) $\sin\theta\sin\alpha=-\frac{a^2}{b^2}$

(B) $\sin\theta\sin\alpha=\frac{a^2}{b^2}$

(C) $\tan\theta\tan\alpha=-\frac{a^2}{b^2}$

(D) $\tan\theta\tan\alpha=\frac{a^2}{b^2}$

9 It is given that $3+i$ is a root of $P(z)=z^3+az^2+bz+10$ where a and b are real numbers. Which expression factorises $P(z)$ over the real numbers?

(A) $(z-1)(z^2+6z-10)$

(B) $(z-1)(z^2-6z-10)$

(C) $(z+1)(z^2+6z+10)$

(D) $(z+1)(z^2-6z+10)$

10 If $x^3 + y^3 x = y^2$, then $\dfrac{dy}{dx}$ is given by:

(A) $\dfrac{3x^2 + y^3}{2y - 3y^2 x}$

(B) $\dfrac{3x^2 + y^3}{3y^2 x - 2y}$

(C) $\dfrac{3x^2 + 3y^2 x + y^3}{2y}$

(D) $\dfrac{3x^2 + y^3}{3y^2 x - 2y}$

End of Section I

Section II

90 Marks
Attempt Questions 11-16
Allow about 2 hours and 40 minutes for this section

Answer each question in a separate writing booklet. Extra writing booklets are available

All necessary working should be shown in every question.

Question 11 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) If $z = 4-2i$ and $w = 3+i$, evaluate $z^2 + \overline{w}$. **2**

(b) (i) Find the Cartesian equation of the locus of z if $\arg\left(\dfrac{z-2}{z}\right) = \dfrac{\pi}{2}$. **1**

(ii) Sketch the locus from part (i) **1**

(c) Find $\displaystyle\int \frac{\sin x}{\cos^3 x}\,dx$ **2**

(d) (i) Express $-\sqrt{3}-i$ in modulus argument form. **2**

(ii) Show that $\left(-\sqrt{3}-i\right)^6$ is a real number **2**

(e) By considering the function $f(x) = \left|x^2-4x\right|$ sketch the graph of $y = \dfrac{1}{f(x)}$ **2**

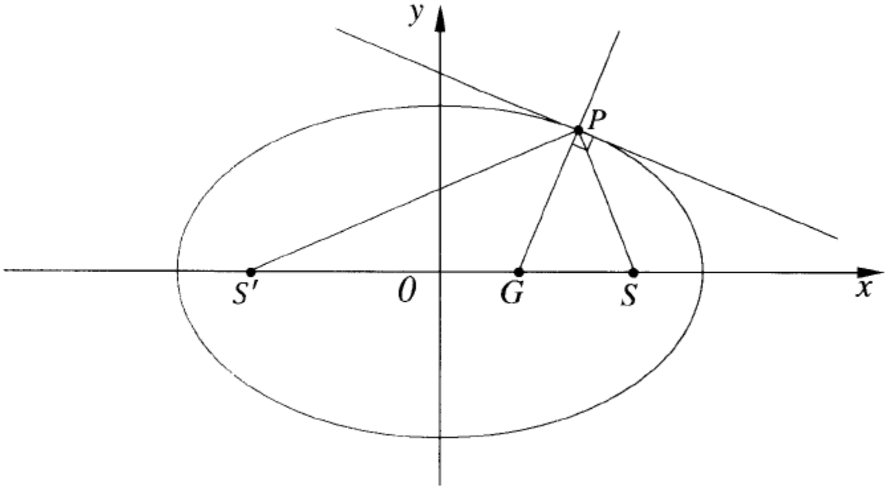
(f) Find $\displaystyle\int \frac{1}{x^2\sqrt{x^2-4}}\,dx$ **3**

End of Question 11

Question 12 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) By using the substitution $t = \tan\left(\frac{x}{2}\right)$, or otherwise, evaluate $\int_0^{\frac{\pi}{2}} \frac{dx}{1+\sin x}$. **4**

(b) The ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ has foci $S(ae,0)$ and $S'(-ae,0)$, and directrices $x = \pm\frac{a}{e}$.
$P(a\cos\theta, b\sin\theta)$ is a point on the ellipse with the normal at P meeting the x-axis at G.



(i) Using the focus/directrix definition of an ellipse show that **2**

$$\frac{PS}{PS'} = \frac{1-e\cos\theta}{1+e\cos\theta}.$$

(ii) The equation of the normal at P is given by **3**

$$\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2.$$ (Do NOT prove this.)

Show that $\frac{GS}{GS'} = \frac{PS}{PS'}$.

(c) Let $I_n = \int_1^e x(\ln x)^n dx$ for n = 0, 1, 2, …, show that $I_n = \frac{e^2}{2} - \frac{n}{2} I_{n-1}$ **3**

(d) Suppose that the complex number z lies on the unit circle, and $0 \le \arg(z) \le \frac{\pi}{2}$. **3**
By the use of a suitable vector diagram, prove that $2\arg(z+1) = \arg(z)$.

End of Question 12

Question 13 **(15 marks)** Use a SEPARATE writing booklet **Marks**

(a) A particle of mass m is thrown vertically upwards with initial velocity U in a medium with resistive force $R = mkv$ where v is the velocity of the particle at time t and k is a constant. The equation of the motion of the particle is then $\frac{dv}{dt} = -g - kv$ where g is the acceleration due to gravity (**Do not** prove this).

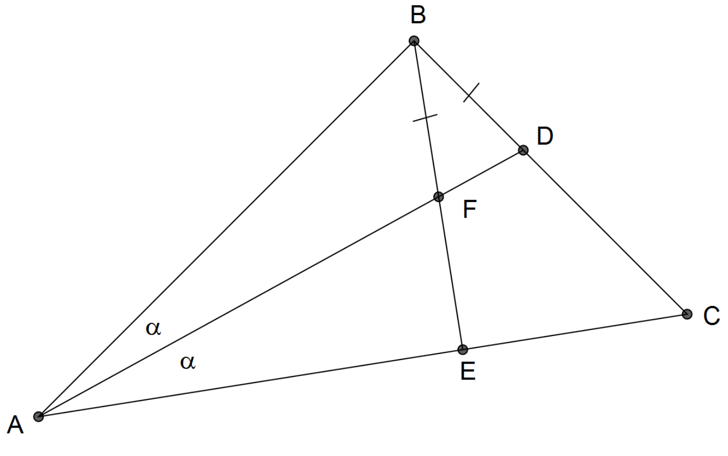
(i) Use $\frac{dv}{dt} = v\frac{dv}{dx}$ to show that the vertical displacement x from the point of projection of the particle is given by **3**

$$x = \frac{1}{k}(U - v) - \frac{g}{k^2}\log_e\left(\frac{g + kU}{g + kv}\right).$$

(ii) Hence find an expression for H the maximum height reached by the particle. **1**

(iii) Find an expression for the time taken for the particle to reach its maximum height. **3**

(b) In the diagram below, AD bisects $\angle BAC$ and F is the point on AD so that BF=BD. Prove that AB, is tangential to the circle passing through B, C and E . **3**



Question 13 is continued on the next page

(c) The hyperbola $\mathcal{H}$ has equation $xy = 16$. The points $P\left(4p, \frac{4}{p}\right)$ for $p > 0$ and $Q\left(4q, \frac{4}{q}\right)$ for $q > 0$ are two distinct arbitrary points on $\mathcal{H}$.

(i) Show that the equation of the tangent at P is $x + p^2 y = 8p$ **1**

(ii) Find the coordinates of T, the point of intersection of the tangents at P and Q. **2**

(iii) The equation of the chord passing through PQ is given by $pqx + y = 4(p+q)$ (**Do not** prove this).

If chord PQ passes through the point $N(0,8)$ find the Cartesian equation of the locus of T **2**

End of Question 13

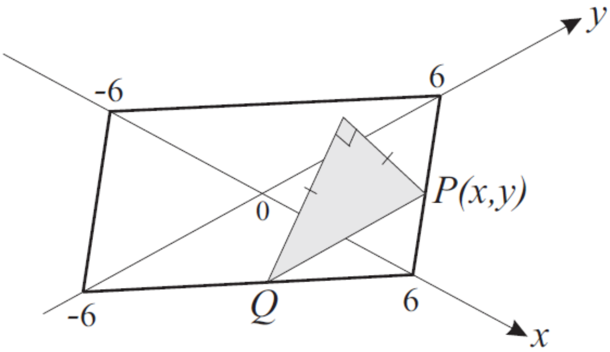
Question 14 **(15 marks)** Use a SEPARATE writing booklet **Marks**

(a) Find $\int \frac{x^2 - 2x - 3}{(x+2)(x^2+1)} dx$ **3**

(b) (i) Draw a one third page sketch the graph of $y = \frac{x^3}{x^2 - 4}$,
indicating the coordinates of all stationary points and all asymptotes. **4**

(ii) For what values of k will $x^3 - kx^2 + 4k = 0$ have exactly one real root. **1**

(c)



The diagram above shows the horizontal square base of a solid.
Vertical cross-sections of the solid perpendicular to the x-axis
are right-angled isosceles triangles with hypotenuse in the base.

(i) Find, as a function of *x*, the area of a typical cross-section standing on the interval *PQ*. **2**

(ii) Find the volume of the solid. **2**

(d) If $U_1 = 8$ and $U_2 = 20$ and $U_n = 4U_{n-1} - 4U_{n-2}$ for $n \geq 3$,
prove by mathematical induction that $U_n = (n+3)2^n$ for $n \geq 1$ **3**

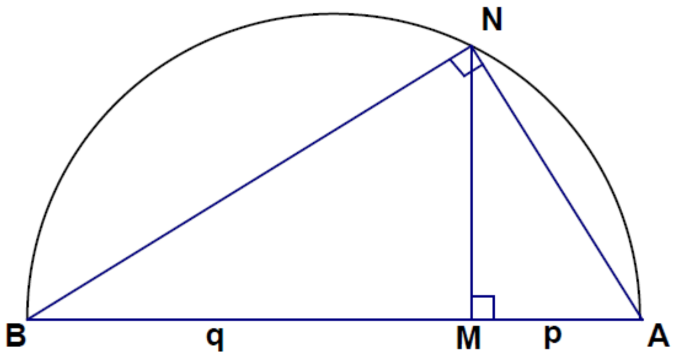
End of Question 14

Question 15 **(15 marks)** Use a SEPARATE writing booklet **Marks**

(a) Show by the use of calculus that $x \geq \ln(x+1)$ for $x > -1$ **3**

Hint: Let $f(x) = x - \ln(x+1)$.

(b) In the diagram, *AB* is the diameter of a semicircle. $\angle ANB = 90^\circ$ and *M* is a point on *AB* such that *NM* is perpendicular to *AB*.



If *AM*=*p* and *BM*=*q*.

(i) Explain why $NM = \sqrt{pq}$ **1**

(ii) By reference to the geometry of the diagram deduce that $\sqrt{pq} \leq \dfrac{p+q}{2}$ **1**

(iii) Hence prove that for $p, q, x, y \geq 0$ then

$$\frac{1}{4}(p+q+x+y) \geq (pqxy)^{\frac{1}{4}}$$ **2**

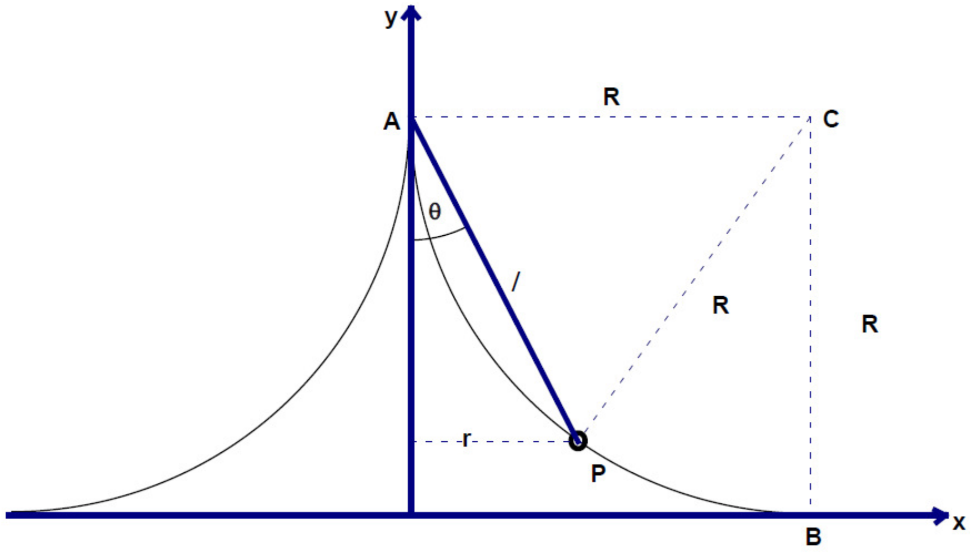
(iv) Deduce that if $k, l, m, n \geq 0$ then $\dfrac{k}{l} + \dfrac{l}{m} + \dfrac{m}{n} + \dfrac{n}{k} \geq 4$ **1**

Question 15 is continued on the next page

Question 15 continued

Marks

(c)



AB is an arc of a circle centre C and radius R. A surface is formed by rotating the arc AB through one revolution about the y-axis. A light, inextensible string of length l, $l \leq R$, is attached to point A, and a particle of mass m is attached to the other end. The particle is set in motion, tracing out a horizontal circle on the surface with constant angular velocity ω radians per second, while the string stays taught.

i) Explain why, when the particle is in position P shown on the diagram, the direction of the force N exerted by the surface on the particle is towards C. **1**

ii) If the string makes an angle θ with the vertical, show that $\angle ACP = 2\theta$. **1**

iii) Show on a diagram the tension force T, the force N and the weight force of magnitude mg acting on the particle, indicating their direction in terms of θ. **1**

iv) Show that **2**

$$T \cos\theta + N \sin 2\theta = mg$$

$$T \sin\theta - N \cos 2\theta = m\, l \sin\theta\ \omega^2$$

v) Show that **1**

$$N = m\, l \sin\theta \left(\frac{g}{l} \sec\theta - \omega^2\right).$$

vi) Deduce that there is a maximum value ω for the motion to occur as described, and write down this maximum value. **1**

End of Question 15

Question 16 **(15 marks)** Use a SEPARATE writing booklet

Marks

(a) A bag contains 10 black and 10 blue marbles. Six marbles are selected without replacement.

(i) Calculate the probability that exactly three marbles selected are blue, giving your answer correct to three decimal places. **1**

(ii) Hence, or otherwise, calculate the probability that more than three of the marbles selected are blue, giving your answer correct to three decimal places. **2**

(b) (i) Find an expression for the limiting sum of infinite geometric series

$1+z+z^2+\ldots$ for $|z|<1$ **1**

(ii) Given that complex number $z=\frac{1}{2}(\cos\theta+i\sin\theta)$, use your answer in part (i) to show that the imaginary part of $1+z+z^2+\ldots$ is $\frac{2\sin\theta}{5-4\cos\theta}$. **3**

(iii) Find an expression for $1+\frac{1}{2}\cos\theta+\frac{1}{2^2}\cos 2\theta+\frac{1}{2^3}\cos 3\theta+\ldots$ in terms of $\cos\theta$ **2**

(c) (i) Find $\lim\limits_{n\to\infty}\left[\tan^{-1}(n+1)+\tan^{-1}(n)\right]$. **1**

(ii) Show that $\tan^{-1}(n+1)-\tan^{-1}(n-1)=\tan^{-1}\left(\frac{2}{n^2}\right)$, where n is a positive integer. **2**

(iii) Hence show that $\lim\limits_{n\to\infty}\sum\limits_{j=1}^{n}\tan^{-1}\left(\frac{2}{j^2}\right)=\frac{3\pi}{4}$ **3**

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

2013 Year 12 Mathematics Extension 2 Trial HSC SOLUTIONS

Suggested Solution (s)	Comments
1. B 2. D 3. A 4. C 5. C 6. B 7. D 8. A 9. D 10 A	
Section I (Solutions)	
1. $\frac{5}{iw} = \frac{5}{i(-2+i)}$ $= \frac{5}{-1-2i} \times \frac{-1+2i}{-1+2i}$ $= -1+2i \quad \therefore B$	
2. $\Delta V = 2\pi xy\cdot\Delta x$ $y = \sqrt{30-x^2} - x^2$ $\therefore V = \int_0^{\sqrt{5}} 2\pi x\left(\sqrt{30-x^2} - x^2\right)dx$ $\therefore D$	
3. By inspection since $f(x) \geq 0 \quad \therefore A$	
4. let $y = x^{\frac{1}{2}}$ $y^{\frac{3}{2}} + 3y = -4 \Rightarrow y^{\frac{3}{2}} = -3y-4$ expanding on squaring gives $y^3 - 9y^2 - 24y - 16 = 0$ $\therefore C$	
5. $\ddot{x} = \frac{6}{x^3} - \frac{10}{x^2}$ $\therefore \frac{1}{2}v^2 = -3x^{-2} + 10x^{-1} + c$ at $v=0,\ x=1 \quad c=-7$ $\therefore v^2 = \frac{2}{x^2}(-3+10x-7x^2)$ $\therefore v = \pm\frac{1}{x}\sqrt{2(-3+10x-7x^2)}$	$\therefore C$
6. $\int \frac{2}{x^2+4x+13}\,dx$ $= \int \frac{2}{x^2+4x+9+4}\,dx$ $= \int \frac{2}{(x+2)^2+3^2}\,dx$ $= \frac{2}{3}\tan^{-1}\left(\frac{x+2}{3}\right) + C$	$\therefore B$
7. $a=4,\ b=3 \quad b^2 = a^2(e^2-1)$ $\therefore e^2 = \frac{25}{16} \quad e = \frac{5}{4}$ $\therefore S(\pm ae, 0) = S(\pm 5, 0) \therefore$	D
8. $m_{OQ} \times m_{OP} = -1$ $\frac{b\tan\alpha}{a\sec\alpha} \times \frac{b\tan\theta}{a\sec\theta} = -1$ $\therefore \sin\alpha\sin\theta = -\frac{a^2}{b^2} \quad \therefore A$	
9. Since $3+i$ is a root $\therefore 3-i$ also a root ($P(z)$ has real co-efficients). $\therefore z^2-6z+10$ is a factor and $\Sigma\alpha\beta\gamma = -10$ so $\therefore D$	
10. $x^3 + y^3x = y^2$ $\therefore 3x^2 + y^3 + x\cdot 3y^2\cdot\frac{dy}{dx} = 2y\cdot$ $\therefore \frac{dy}{dx} = \frac{3x^2+y^3}{2y-3xy^2}$ $\therefore A$	$\frac{dy}{dx}$

2013 Year 12 Mathematics Extension 2 Trial HSC SOLUTIONS

Suggested Solution (s)	Comments	Suggested Solution (s)	Comments

Question 11

(a) $z^2 + \bar{w} = (4-2i)^2 + 3 - i$

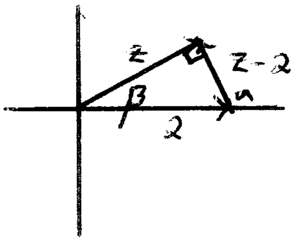
$= 12 - 16i + 3 - i$ ✓

$= 15 - 17i$ ✓

(b) $\arg(z-2) - \arg(z) = \frac{\pi}{2}$

let $\alpha - \beta = \frac{\pi}{2}$

$\therefore \alpha = \beta + \frac{\pi}{2}$



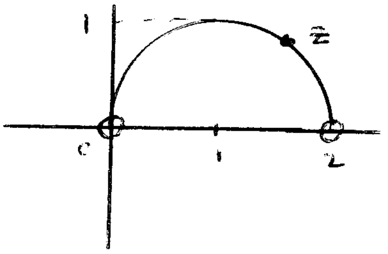
i) So locus is semi circle centre $(1,0)$, radius 1 unit

$y = \sqrt{1-(x-1)^2}$

$y = \sqrt{2x - x^2}$, $0 < y \le 1$ ✓

or $(x-1)^2 + y^2 = 1$, $0 < y \le 1$

ii)

 ✓

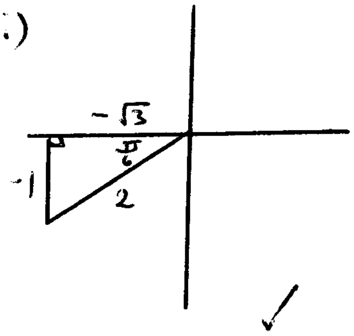
c) $\int \sin x (\cos x)^{-3} dx$

$= -\int -\sin x (\cos x)^{-3} dx$ ✓

$= \frac{1}{2}(\cos x)^{-2} + C$

$= \frac{1}{2}\sec^2 x + C$ ✓

d) i)



✓ ✓

$\therefore -\sqrt{3} - i = 2\,\text{cis}\left(\frac{7\pi}{6}\right)$

or equivalent.

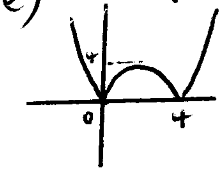
ii) $(-\sqrt{3} - i)^6 = \left[2\,\text{cis}\left(\frac{7\pi}{6}\right)\right]^6$

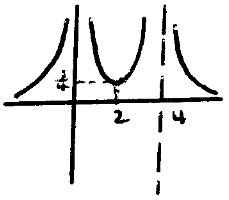
$= 2^6\,\text{cis}(7\pi)$ by DMT ✓

$= 2^6 \times -1$

$= -64$ $\therefore$ Real ✓

e) ✓ ✓





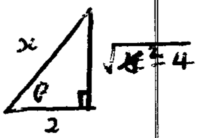
$y = |x(x-4)|$

f) let $x = 2\sec\theta$ $\therefore dx = 2\sec\theta\tan\theta\, d\theta$ ✓

$\therefore \int \frac{1}{4\sec^2\theta\sqrt{4\sec^2\theta - 4}} \cdot 2\sec\theta\tan\theta$

$= \frac{1}{2}\int \frac{\tan\theta}{\sec\theta\; 2\tan\theta} \cdot d\theta$

$= \frac{1}{4}\int \cos\theta\, d\theta$ ✓



$= \frac{1}{4}\sin\theta + C$

$= \frac{\sqrt{x^2-4}}{4x} + C$ ✓

2013 Year 12 Mathematics Extension 2 Trial HSC SOLUTIONS

Suggested Solution (s) | **Comments**

Question 12

let $I = \int_0^{\frac{\pi}{2}} \frac{1}{1+\sin x} \cdot dx$

and let $t = \tan(\frac{x}{2})$

$\therefore dx = \frac{2dt}{1+t^2}$

and $x = 0, \; t = 0$
$x = \frac{\pi}{2}, \; t = 1$ ✓

$\therefore I = \int_0^1 \frac{1}{1+\frac{2t}{1+t^2}} \cdot \frac{2\,dt}{1+t^2}$ ✓

$= \int_0^1 \frac{2}{t^2+2t+1}\,dt$ ✓

$= \int_0^1 2(t+1)^{-2}\,dt$

$= -2\Big[(t+1)^{-1}\Big]_0^1$

$= -2\left(\frac{1}{2} - 1\right)$

$= 1$ ✓

b) i) Let M and M' be the feet of the perpendiculars from P to the directrices $x = \frac{a}{e}$ and $x = -\frac{a}{e}$

Since $PS = ePM$
$= e\left(\frac{a}{e} - a\cos\theta\right)$
$= a(1 - e\cos\theta)$ ✓

Suggested Solution (s) | **Comments**

and $PS' = ePM'$
$= e\left(a\cos\theta + \frac{a}{e}\right)$
$= a(1 + e\cos\theta)$ ✓

$\therefore \frac{PS}{PS'} = \frac{1 - e\cos\theta}{1 + e\cos\theta}$

ii) Normal at P meets x-axis at G.

at $y = 0$, $x = \frac{(a^2-b^2)\cos\theta}{a}$

and since $b^2 = a^2(1-e^2)$

$x = \frac{a^2e^2\cos\theta}{a}$
$= ae^2\cos\theta$ ✓

$\therefore G(ae^2\cos\theta, 0)$ with $S(ae, 0)$ & $S'(-ae, 0)$

$\therefore GS = ae - ae^2\cos\theta$
and $GS' = ae + ae^2\cos\theta$ ✓

So $\frac{GS}{GS'} = \frac{ae - ae^2\cos\theta}{ae + ae^2\cos\theta}$
$= \frac{1 - e\cos\theta}{1 + e\cos\theta}$ ✓
$= \frac{PS}{PS'}$ as required

c) $I_n = \int_1^e x(\ln x)^n\,dx$ ✓, $u = (\ln x)^n$, $u' = \frac{n(\ln x)^{n-1}}{x}$ ✓, $v' = x$, $v = \frac{x^2}{2}$

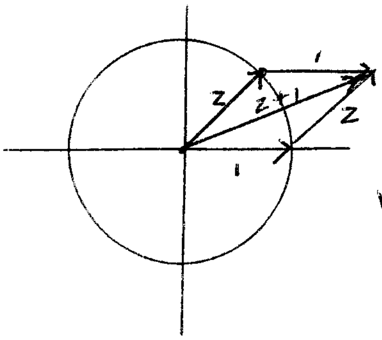
$= \left.\frac{x^2(\ln x)^n}{2}\right]_1^e - \frac{1}{2}n\int_1^e x(\ln x)^{n-1}\,dx$ ✓

$= \frac{e^2(\ln e)^n}{2} - 0 - \frac{1}{2}n\int_1^e x(\ln x)^{n-1}\,dx$

$= \frac{e^2}{2} - \frac{n}{2} I_{n-1}$

Suggested Solution (s) **Comments**

d)



✓

let $\arg(z) = \theta$

$\arg(z+1) = \frac{\theta}{2}$

(diagonal of rhombus bisects the angle) ✓

So $2\arg(z+1) = \theta$ ✓

and equating

$2\arg(z+1) = \arg(z)$

Question 13

a) i) Let $v\frac{dv}{dx} = -g - kv$

$$\frac{dx}{dv} = \frac{-v}{g+kv}$$

so $x = \int \frac{-v}{g+kv}\,dv$ ✓

$$= -\frac{1}{k}\int \frac{g+kv-g}{g+kv}\,dv$$

$$= -\frac{1}{k}\int 1 - \frac{g}{g+kv}\,dv$$

$$= -\frac{1}{k}\int 1\,dv + \frac{g}{k^2}\int \frac{k}{g+kv}\,dv$$ ✓

$$= -\frac{v}{k} + \frac{g}{k^2}\ln(g+kv) + C$$ ✓

at $x=0$ $v=U$

$\therefore C = \frac{U}{k} - \frac{g}{k^2}\ln(g+kU)$ ✓

$$\therefore x = \frac{U}{k} - \frac{v}{k} - \frac{g}{k^2}\left[\ln(g+kU) - \ln(g+kv)\right]$$

$$\therefore x = \frac{1}{k}(U-v) - \frac{g}{k^2}\left[\ln\left(\frac{g+kU}{g+kv}\right)\right]$$

as required.

ii) Max Height $x=H$ when $v=0$

$\therefore x = \frac{1}{k}U - \frac{g}{k^2}\left[\ln\left(\frac{g+kU}{g}\right)\right]$ ✓

iii) $\frac{dv}{dt} = -g - kv$

$$\therefore \frac{dt}{dv} = \frac{-1}{g+kv}$$

$$\therefore t = \int \frac{-1}{g+kv}\,dv$$

$$= -\frac{1}{k}\int \frac{k}{g+kv}\,dv$$

so $t = -\frac{1}{k}\ln(g+kv) + c$ ✓

at $t=0$, $v=U$ $\therefore c = \frac{1}{k}\ln(g+kU)$

$\therefore t = \frac{1}{k}\ln\left(\frac{g+kU}{g+kv}\right)$ ✓

at Max height $v=0$

$\therefore T = \frac{1}{k}\ln\left(\frac{g+kU}{g}\right)$ ✓

Suggested Solution (s) | **Comments**

b) For AB to be tangential to circle thru BCE, need to show $\angle ABE = \angle BCE$.

Let $\angle ABF = \beta$

$\therefore \angle BFD = \alpha + \beta$ (Ext $\angle$ of $\triangle ABF$)

$= \angle BDF$ ($BF = BD$ given)

$\therefore \angle FBD = \pi - (2\alpha + 2\beta)$ ($\angle$ sum of $\triangle$) ✓

Also $\angle AFE = \alpha + \beta$ (Vert. opp).

$\therefore \angle FED = 2\alpha + \beta$ (Ext. $\angle$ of $\triangle FAE$)

$\therefore \angle BCE = \pi - (\pi - (2\alpha + 2\beta)) - (\alpha + \beta)$

$= \pi - \pi + 2\alpha + 2\beta - 2\alpha - \beta$

$= \beta$ ✓

$\therefore \angle ABE = \angle BCE$

so AB is tangential as angle made between tangent and chord BE is equal to angle in the alternate segment. ✓

Suggested Solution (s) | **Comments**

c) i) $xy = 16$

$\frac{d}{dx}(xy) = 0$

$x\frac{dy}{dx} + y = 0$

$\therefore \frac{dy}{dx} = \frac{-y}{x}$

at P $m_T = \frac{-\frac{4}{p}}{4p}$

$= -\frac{1}{p^2}$

$\therefore y - \frac{4}{p} = -\frac{1}{p^2}(x - 4p)$

$p^2 y - 4p = -x + 4p$ ✓

$\therefore x + p^2 y = 8p$

ii) Similarly $x + q^2 y = 8q$ for tangent at Q.

subtracting.

$(p^2 - q^2)y = 8(p - q)$

$\therefore y = \frac{8}{p+q}$ ✓

so $x + \frac{8p^2}{p+q} = 8p$

$x = \frac{8p^2 + 8pq - 8p^2}{p+q}$ ✓

$= \frac{8pq}{p+q}$

$\therefore T\left(\frac{8pq}{p+q}, \frac{8}{p+q}\right)$

Suggested Solution (s) | **Comments**

iii) $N(0,8)$ $p+q=2$

for T $x = \frac{8pq}{p+q}$

and $y = \frac{8}{p+q}$ ✓

so $y=4$

but if $p,q>0$

then $x = \frac{8pq}{p+q} > 0$

and since $p+q=2$ $pq = 2p-p^2$ has max value of 1

∴ Locus of $y=4$, $0<x\leq4$ ✓

Question 14

a) let $\frac{x^2-2x-3}{(x+2)(x^2+1)} \equiv \frac{a}{x+2} + \frac{bx+c}{x^2+1}$

∴ $x^2-2x-3 \equiv a(x^2+1)+(bx+c)(x+2)$

let $x=-2$ ∴ $5=5a \Rightarrow a=1$

let $x=0$ ∴ $-3=1+2c \Rightarrow c=-2$

let $x=1$ ∴ $-4=2+(b-2)3$

$-6=3b-6$

$b=0$

✓✓

∴ let $I = \int \frac{1}{x+2}\,dx + \int \frac{-2}{x^2+1}\,dx$

$= \ln|x+2| - 2\tan^{-1}(x) + C$ ✓

Suggested Solution (s) | **Comments**

b) i) $y = \frac{x^3}{x^2-4}$

$$x^2-4\,\overline{)\,x^3+0x\,}\quad \text{quotient } x;\quad x^3-4x;\quad +4x$$

$= x + \frac{4x}{x^2-4} = x + \frac{4x}{(x-2)(x+2)}$

$y' = \frac{(x^2-4)\cdot 3x^2 - x^3(2x)}{(x^2-4)^2}$

$= \frac{3x^4-12x^2-2x^4}{(x^2-4)^2}$

$= \frac{x^4-12x^2}{(x^2-4)^2} = 0$

when $x^2=0$ $=12$

$x=0$ $x=\pm 2\sqrt{3}$

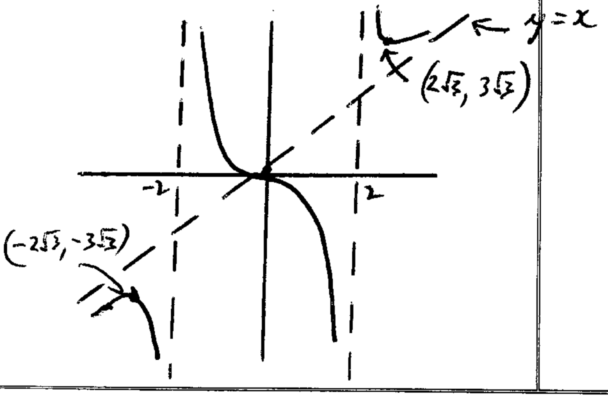
$y = \frac{(2\sqrt{3})^3}{8} = \frac{8\times 3\sqrt{3}}{8} = 3\sqrt{3}$

∴ Stat Pts $(\pm2\sqrt{3},\pm3\sqrt{3})$ $(0,0)$

Asymptotes: $x \neq \pm 2$

as $x \to \infty$, $y \to x^+$

Odd function.



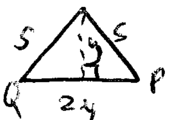
Comments:

Stat pts ✓

Asymptotes ✓ $x=\pm2$

Asymptote $y=x$ ✓

Shape ✓

Suggested Solution (s)	Comments	Suggested Solution (s)	Comments
ii) Consider when $\frac{x^3}{x^2-4} = k$ has one sol$\underline{n}$. by inspection of graph: $-3\sqrt{3} < k < 3\sqrt{3}$ ✓ c) i)  $\therefore s = \sqrt{2}y$ $\therefore$ Area $= \frac{1}{2} \times 2y \times y$ ✓ $= y^2$ $PQ = (6-x) - (x-6)$ $= 12 - 2x$ $= 2(6-x)$ $\therefore$ Area $= (6-x)^2$ ✓ ii) ✓ $\therefore V = 2\int_0^6 (6-x)^2\,dx$ ✓ (symmetry of solid) $= 2\left[\frac{(6-x)^3}{-3}\right]_0^6$ $= 2\left(0 - \left(\frac{6^3}{-3}\right)\right)$ $= 144\ u^3$ ✓	note $PQ = 2(6+x)$ in 2nd/3rd quadrants.	d) When $n=1$, $U_1 = 4 \times 2^1$ $= 8$ and $n=2$ $U_2 = 5 \times 2^2$ $= 20$ $\therefore$ Statement True for $n=1,2$ Assume statement true for $n=k, n=k-1$ ie $U_{k-1} = (k+2)\cdot 2^{k-1}$ $U_k = (k+3)\cdot 2^k$ When $n=k+1$ $U_n = U_{k+1}$ $= 4U_{(k+1)-1} - 4U_{(k+1)-2}$ $= 4U_k - 4U_{k-1}$ $= 4(k+3)2^k - 4(k+2)2^{k-1}$ $= (4k+12)\cdot 2^k - 2(k+2)\cdot 2^k$ $= 2k\cdot 2^k - 8\cdot 2^k$ $= k\cdot 2^{k+1} + 4\cdot 2^{k+1}$ $= (k+4)\cdot 2^{k+1}$ $= (n+3)\cdot 2^n$ for $n = k+1$	✓ ✓ ✓

Suggested Solution (s) **Comments**

Question 15

(a) $f(x) = x - \ln(x+1)$

$f'(x) = 1 - \frac{1}{x+1}$

$f''(x) = \frac{1}{(x+1)^2}$ ✓

Noting $f'(0) = 0$ $f''(0) > 0$

$\therefore$ $(0,0)$ is a min. stat. pt. ✓

Noting $f''(x) \geq 0$ for all $x > -1$ ✓

$\therefore$ $f(x)$ concave up for all $x > -1$

$\therefore$ $f(x) \geq 0$

$\therefore$ $x \geq \ln(1+x)$

(b) i) Since $\Delta BMN \,|||\, \Delta NMA$

$\therefore$ $\frac{BM}{NM} = \frac{NM}{MA}$ ✓

$NM^2 = AM.MB$

$= pq$

$\therefore$ $NM = \sqrt{pq}$

ii) $AB = p+q$ is diameter

$\therefore$ $\frac{p+q}{2}$ is radius

Since $NM \perp AB$, MN is always less than length of radius ✓

$\therefore$ $MN \leq \frac{p+q}{2}$

$\therefore$ $\sqrt{pq} \leq \frac{p+q}{2}$

Suggested Solution (s) **Comments**

iii) $\frac{1}{4}(p+q+x+y)$

$= \frac{1}{2}\left(\frac{p+q}{2} + \frac{x+y}{2}\right)$ ✓

$\geq \frac{1}{2}\left(\sqrt{pq} + \sqrt{xy}\right)$

$\geq \sqrt{\sqrt{pq} \cdot \sqrt{xy}}$ ✓

$\geq (pqxy)^{\frac{1}{4}}$

iv) Replacing p, q, x, y with $\frac{k}{l}, \frac{l}{m}, \frac{m}{n}, \frac{n}{k}$ ✓

$\frac{1}{4}\left(\frac{k}{l} + \frac{l}{m} + \frac{m}{n} + \frac{n}{k}\right) \geq \left(\frac{k}{l} \cdot \frac{l}{m} \cdot \frac{m}{n} \cdot \frac{n}{k}\right)^{\frac{1}{4}}$

$\therefore$ $\frac{k}{l} + \frac{l}{m} + \frac{m}{n} + \frac{n}{k} \geq 4$

(c) (i) The normal force is at right angles to the tangent to the circle and is therefore directed towards the circle's centre C. ✓

ii) In ΔACP,

$\angle PAC = \frac{\pi}{2} - \theta$ (complimentary, y-axis tangent to circle).

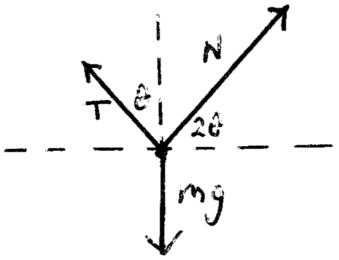
$= \angle APC$ ($AC = AP$, equal radii)

$\therefore$ $\angle ACP = \pi - 2\left(\frac{\pi}{2} - \theta\right)$ ($\angle$ sum of Δ) ✓

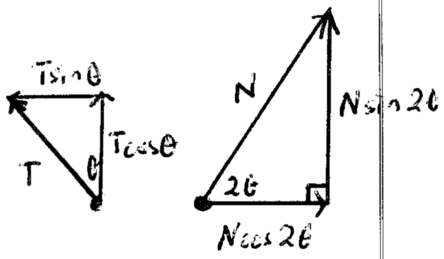
$= 2\theta$.

Suggested Solution (s) | **Comments** | **Suggested Solution (s)** | **Comments**

iii)



iv)



Vertically forces are balanced ✓

∴ $T\cos\theta + N\sin 2\theta = mg$ — ①

Sum of radial forces = $mr\omega^2$ towards centre of motion.

$T\sin\theta - N\cos 2\theta = mr\omega^2$ ✓

but $\sin\theta = \frac{r}{l}$ ∴ $r = l\sin\theta$

∴ $T\sin\theta = ml\sin\theta\,\omega^2$ — ②

v) ① × sinθ − ② × cosθ gives

$N\sin 2\theta\sin\theta + N\cos 2\theta\cos\theta = mg\sin\theta - ml\sin\theta\cos\theta\,\omega^2$

∴ $N\cos(2\theta - \theta) = m\sin\theta(g - ml\cos\theta\,\omega^2)$ ✓

∴ $N\cos\theta = ml\sin\theta\cos\theta\left(\frac{g\sec\theta}{l} - \omega^2\right)$

∴ $N = ml\sin\theta\left(\frac{g}{l}\sec\theta - \omega^2\right)$

vi) For the motion $N \geq 0$

∴ $\frac{g\sec\theta}{l} - \omega^2 \geq 0$

$\omega \leq \sqrt{\frac{g}{l}\sec\theta}$

∴ Max is $\sqrt{\frac{g}{l}\sec\theta}$ ✓

Question 16

(a) i) $P(3\text{ blue}) = \frac{{}^{10}C_3 \times {}^{10}C_3}{{}^{20}C_6}$

$= 0.372$ (3 d.p).

ii) $P(>3\text{ blue})$

$= P(4\text{ blue}) + P(5\text{ blue}) + P(6\text{ blue})$ ✓

or

$= \frac{1 - P(3\text{ blue})}{2}$ (symmetry)

$= 0.314$ (3 d.p.) ✓

(b) i)

$S_\infty = \frac{1}{1-z}$ ✓

Suggested Solution (s)	Comments	Suggested Solution (s)	Comments

Question 16 (continued)

ii) $\frac{1}{1-z} = \frac{1}{1-\left(\frac{1}{2}\cos\theta + \frac{1}{2}i\sin\theta\right)}$

$= \frac{1}{\left(1-\frac{1}{2}\cos\theta\right) - \frac{1}{2}i\sin\theta} \times \frac{1-\frac{1}{2}\cos\theta + \frac{1}{2}i\sin\theta}{1-\frac{1}{2}\cos\theta + \frac{1}{2}i\sin\theta}$ ✓

$= \frac{1-\frac{1}{2}\cos\theta + \frac{1}{2}i\sin\theta}{\left(1-\frac{1}{2}\cos\theta\right)^2 + \frac{1}{4}\sin^2\theta}$

$= \frac{1-\frac{1}{2}\cos\theta + \frac{1}{2}i\sin\theta}{1-\cos\theta + \frac{1}{4}\cos^2\theta + \frac{1}{4}\sin^2\theta}$

$= \frac{1-\frac{1}{2}\cos\theta + \frac{1}{2}i\sin\theta}{\frac{5}{4}-\cos\theta}$ ✓

$\therefore \operatorname{Im}\left(\frac{1}{1-z}\right) = \frac{\frac{1}{2}\sin\theta}{\frac{5}{4}-\cos\theta} \times \frac{4}{4}$ ✓

$= \frac{2\sin\theta}{5-4\cos\theta}$

iii) $1 + \frac{1}{2}\cos\theta + \frac{1}{2^2}\cos 2\theta + \frac{1}{3^2}\cos 3\theta + \dots$ is real part of $\frac{1}{1-z}$ by DMT. ✓

$\therefore \operatorname{Re}\left(\frac{1}{1-z}\right) = \frac{1-\frac{1}{2}\cos\theta}{\frac{5}{4}-\cos\theta} \times \frac{4}{4}$

$= \frac{4-2\cos\theta}{5-4\cos\theta}$ ✓

(c) i) π ✓

ii) let $\alpha = \tan^{-1}(n+1)$

and $\beta = \tan^{-1}(n-1)$

$\therefore \tan(\alpha-\beta)$ ✓

$= \frac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta}$

$= \frac{(n+1)-(n-1)}{1+(n-1)(n+1)} = \frac{2}{n^2}$ ✓

$\therefore \alpha - \beta = \tan^{-1}\left(\frac{2}{n^2}\right)$

iii) $\sum_{j=1}^{n} \tan^{-1}\left(\frac{2}{j^2}\right)$

$= \tan^{-1}\left(\frac{2}{1^2}\right) + \tan^{-1}\left(\frac{2}{2^2}\right) + \tan^{-1}\left(\frac{2}{3^2}\right) + \dots + \tan^{-1}\left(\frac{2}{(n-1)^2}\right) + \tan^{-1}\left(\frac{2}{n^2}\right)$ ✓

$= \tan^{-1}(2) - \tan^{-1}(0)$

$+ \cancel{\tan^{-1}(3)} - \tan^{-1}(1)$

$+ \tan^{-1}(4) - \cancel{\tan^{-1}(2)}$

$+ \cancel{\tan^{-1}(5)} - \cancel{\tan^{-1}(3)}$

$+ \dots$

$+ \tan^{-1}(n-1) - \cancel{\tan^{-1}(n-3)}$

$+ \tan^{-1}(n) - \tan^{-1}(n-2)$

$+ \tan^{-1}(n+1) - \cancel{\tan^{-1}(n-1)}$ ✓

✓

$= -\tan^{-1}(0) - \tan^{-1}(1) + \tan^{-1}(n+1) + \tan^{-1}(n)$

$= 0 - \frac{\pi}{4} + \pi$ as $n \to \infty$ from part (i)

$\lim_{n\to\infty} \sum_{j=1}^{n} \tan^{-1}\left(\frac{2}{j^2}\right) = \frac{3\pi}{4}$